

# Triangle Order $\leq_{\Delta}$ in Singular Categories

Zhengfang WANG\*

## Abstract

Degeneration of modules is defined geometrically. Riedtmann and Zwara show that this degeneration is equivalent to the existence of a certain short exact sequence. Then Yoshino and independently Jensen, Su and Zimmermann generalised this notion to triangulated categories. We write  $X \leq_{\Delta} Y$  if  $X$  degenerates to  $Y$ . In this paper, we prove that  $\leq_{\Delta}$  applied to the singular category  $\mathcal{D}_{\text{sg}}(A)$  of a finite-dimensional  $k$ -algebra  $A$  induces a partial order on the set of isomorphism classes of objects in  $\mathcal{D}_{\text{sg}}(A)$ .

## 1 Introduction

Degeneration order of modules is introduced from geometric methods of representation theory of finite dimensional algebras. More precisely, let  $A$  be a finite dimensional associative  $k$ -algebra over the algebraically closed field  $k$ . Let  $d$  be a positive integer. A  $d$ -dimensional (left)  $A$ -module  $M$  is the vector space  $k^d$  together with an action by  $A$  from the left. We denote by  $\text{mod}_d(A)$  the set of  $d$  dimensional  $A$ -modules. Note that  $\text{mod}_d(A)$  is an affine variety (For more ample details we refer to Section 2). The general linear group  $\text{GL}_d(k)$  acts on  $\text{mod}_d(A)$  by conjugation. The orbits under this action are the isomorphism classes of  $d$ -dimensional  $A$ -modules. We say that an  $A$ -module  $N$  is called a degeneration of  $M$  (denote by  $M \leq_{\text{deg}} N$ ) if  $N$  belongs to the Zariski closure of the  $\text{GL}_d(k)$ -orbit of  $M$  in  $\text{mod}_d(A)$ . Clearly, this degeneration defines a partial order on the set of isomorphism classes of  $A$ -modules. Riedtmann and Zwara gave an algebraic description of degeneration order in [Rie] and [Zwa2]. They showed that  $M \leq_{\text{deg}} N$  if and only if there is an  $A$ -module  $Z$  and an exact sequence

$$0 \rightarrow N \rightarrow M \oplus Z \rightarrow Z \rightarrow 0$$

and equivalently there exist an  $A$ -module  $Z'$  and an exact sequence

$$0 \rightarrow Z' \rightarrow Z' \oplus M \rightarrow N \rightarrow 0.$$

Later in [Yosh2] Yoshino gave a scheme-theoretical definition of degenerations, so that it can be considered for modules of a Noetherian algebra. In this paper, we consider the algebraic description of degeneration order as our definition of degeneration order of  $A\text{-mod}$ . That

---

\*zhengfang.wang@imj-prg.fr, Université Paris Diderot-Paris 7, Institut de Mathématiques de Jussieu-Paris Rive Gauche CNRS UMR 7586, Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France

is, let  $A$  be a finite dimensional  $k$ -algebra  $A$  over any field  $k$  (not necessary algebraically closed), we define an  $A$ -module  $N$  is a degeneration of an  $A$ -module  $M$  (still denoted by  $\leq_{\text{deg}}$ ) if there exists an  $A$ -module  $Z$  and an exact sequence

$$0 \rightarrow N \rightarrow M \oplus Z \rightarrow Z \rightarrow 0.$$

Then from Theorem 2.2 in [Zwa1] it follows that this degeneration  $\leq_{\text{deg}}$  is a partial order on the set of isomorphism classes of  $A$ -modules.

Degeneration theory for triangulated categories and derived categories (cf. [JSZ1], [JSZ2], [SaZi]) has been studied. In a triangulated category we say for two objects  $X$  and  $Y$  that  $X \leq_{\Delta} Y$  if there is an object  $Z$  and a distinguished triangle

$$Z[-1] \rightarrow N \rightarrow Z \oplus M \rightarrow Z.$$

In [JSZ1], Jensen, Su and Zimmermann showed that the triangle relation  $\leq_{\Delta}$  in bounded derived category  $\mathcal{D}^b(A)$  is a partial order. More generally, in [JSZ2], the authors showed that, under some finiteness assumptions on the triangulated category including the condition that the morphism spaces between objects are finite dimensional,  $\leq_{\Delta}$  is always a partial order.

The singular category was introduced by Ragnar-Olaf Buchweitz in an unpublished manuscript [Buch], he called it stable derived category, and Dmitri Orlov [Orl] rediscovered this notion independently in algebraic geometry and mathematical physics, under the name of singular category. We remark that in general the singular category  $\mathcal{D}_{\text{sg}}(A)$  of a finite-dimensional  $k$ -algebra  $A$  is not Hom-finite and is in general not a Krull-Schmidt category (cf. [Chen], [ZhZi]). Therefore we cannot use [JSZ2] to argue that the triangle relation  $\leq_{\Delta}$  in  $\mathcal{D}_{\text{sg}}(A)$  for any finite dimensional  $k$ -algebra  $A$  is a partial order. But in this paper, we will prove, in a different way, that  $\leq_{\Delta}$  in  $\mathcal{D}_{\text{sg}}(A)$  is really a partial order for any finite dimensional  $k$ -algebra  $A$ .

This paper is organised as follows, in Section 2, we define a stable degeneration  $\leq_{\text{st}}$  for the stable module category  $A\text{-}\underline{\text{mod}}$  and prove that it is a partial order. In Section 3, we first recall some notions about the stabilization  $\mathcal{S}(A\text{-}\underline{\text{mod}})$  of the left triangulated category  $A\text{-}\underline{\text{mod}}$  and prove that in  $\mathcal{S}(A\text{-}\underline{\text{mod}})$ , the triangle relation  $\leq_{\Delta}$  coincides with the quasi-stable degeneration  $\leq_{\text{qst}}$  induced from the stable degeneration order  $\leq_{\text{st}}$  in  $A\text{-}\underline{\text{mod}}$ . We prove that the quasi-stable degeneration  $\leq_{\text{qst}}$  in  $\mathcal{S}(A\text{-}\underline{\text{mod}})$  is a partial order. Last we show our main result (Theorem 3.7), that is, the triangle order  $\leq_{\Delta}$  in  $\mathcal{D}_{\text{sg}}(A)$  is a partial order. We note that Theorem 3.7 and its proof extend without any changes to finitely generated modules over artinian algebras.

For the background on the degeneration theory of modules and triangulated categories, we refer to [JSZ1], [JSZ2], [Rie], [SaZi], [Yosh1], [Yosh2], [Zwa1] and [Zwa2]. For the definition of singular category, we refer to [Buch], [Orl] and [Zim].

**Acknowledgements:** This work is a part of author's PhD thesis, I would like to thank my PhD supervisor Alexander Zimmermann for his many valuable suggestions for improvement. I also want to thank Guodong Zhou for many inspiring conversations.

## 2 Stable degeneration order

First, let us recall the geometrical definition of degeneration order. Let  $A$  be a finite dimensional associative  $k$ -algebra over an algebraically closed field  $k$ . Let  $n$  be the dimension of  $A$  and  $d$  be a positive integer. Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be a  $k$ -basis of  $A$ ,  $\lambda_i \lambda_j = \sum_l a_{ij}^l \lambda_l$  for  $i, j = 1, \dots, n$  with the structure constants  $a_{ij}^l \in k$ . Then a module  $M$  corresponds to a unique  $n$ -tuple of matrices  $m = (m_1, \dots, m_n) \in (\text{Mat}_{d \times d}(k))^n$  such that  $m_i m_j = \sum_l a_{ij}^l m_l$  for  $i, j = 1, \dots, n$ . For each  $1 \leq l \leq n$  let  $X^l$  denote the indeterminate matrix  $(X_{\mu\nu}^l)_{\mu, \nu=1, \dots, d}$ . Then there is a one-to-one correspondence between  $\text{mod}_d(A)$  and the zero set of the ideal  $I \subset k[x_{\mu\nu}^l]$ ,  $(\mu, \nu = 1, \dots, d; l = 1, \dots, n)$ , where  $I$  is generated by the equations of the matrices  $X^i X^j - \sum_l a_{ij}^l X^l$  for  $i, j = 1, \dots, n$ . The general linear group  $\text{GL}_d(k)$  acts on  $\text{mod}_d(A)$  by conjugation. The orbits under this action are the isomorphism classes of  $d$ -dimensional  $A$ -modules. We say that an  $A$ -module  $N$  is called a degeneration of  $M$  (denote by  $M \leq_{\text{deg}} N$ ) if  $N$  belongs to the Zariski closure of the  $\text{GL}_d(k)$ -orbit of  $M$  in  $\text{mod}_d(A)$ . Clearly, this degeneration defines a partial order on the set of isomorphism classes of  $A$ -mod. From the work of Riedtmann and Zwara in [Rie] and [Zwa2], we have an algebraic description of degeneration order. In this paper, we use this algebraic description of degeneration order as our definition of degeneration order.

**Definition 2.1.** Let  $k$  be a field. Let  $A$  be a finite dimensional  $k$ -algebra and  $X, Y \in A\text{-mod}$ , then we say that  $X \leq_{\text{deg}} Y$  if there exist  $Z \in A\text{-mod}$  and an exact sequence on  $A\text{-mod}$ ,

$$0 \rightarrow Y \rightarrow X \oplus Z \rightarrow Z \rightarrow 0.$$

**Remark 2.2.** From Theorem 2.3 in [Zwa1],  $X \leq_{\text{deg}} Y$  is equivalent to that there exists a short exact sequence

$$0 \rightarrow Z' \rightarrow Z' \oplus X \rightarrow Y \rightarrow 0$$

for some  $A$ -module  $Z'$ . We remark that the degeneration order  $\leq_{\text{deg}}$  in  $A\text{-mod}$  defines a partial order on the set of isomorphism classes of  $A$ -modules (cf. [Yosh2], [Zwa1]).

**Definition 2.3.** Let  $A$  be a finite-dimensional  $k$ -algebra and  $X, Y \in A\text{-mod}$ . We say that  $X \leq_{\text{st}} Y$  if and only if there exist two projective  $A$ -modules  $P$  and  $Q$  such that  $X \oplus P \leq_{\text{deg}} Y \oplus Q$ . Clearly, we can induce a relation (called stable degeneration, still denote by  $\leq_{\text{st}}$ ) on isomorphism classes of objects in  $A\text{-mod}$ .

**Remark 2.4.** Note that  $P = 0$  in  $A\text{-mod}$  if and only if  $P$  is a projective module, so that  $X \leq_{\text{st}} Y$  is well-defined for any two objects  $X$  and  $Y$  in  $A\text{-mod}$ .

**Lemma 2.5.** Let  $A$  be a finite dimensional  $k$ -algebra. Then  $\leq_{\text{st}}$  defines a partial order on the set of isomorphism classes in  $A\text{-mod}$ .

*Proof.* First, it's clear that the reflexivity is inherited from the degeneration order. We need to check anti-symmetry and transitivity. For anti-symmetry, let  $X \leq_{\text{st}} Y$  and  $Y \leq_{\text{st}} X$ . Therefore there exist projective  $A$ -modules  $P, Q, P', Q'$  such that

$$X \oplus P \leq_{\text{deg}} Y \oplus Q$$

and

$$Y \oplus P' \leq_{\deg} X \oplus Q'.$$

From the transitivity of  $\leq_{\deg}$ , we have

$$X \oplus P \oplus P' \leq_{\deg} X \oplus Q \oplus Q'.$$

Therefore we obtain (cf. Proposition 4.4 [Yosh1])

$$\dim_k \operatorname{Hom}_A(P \oplus P', S) \leq \dim_k \operatorname{Hom}_A(Q \oplus Q', S) \quad (1)$$

for any simple  $A$ -module  $S$ . Note that there exists a canonical bijection between the isomorphism classes of projective indecomposable modules and the isomorphism classes of simple modules for a finite dimensional  $k$ -algebra  $A$  (cf. e.g. [Lein]), hence it follows from the inequality (1) that  $P \oplus P'$  is a direct summand of  $Q \oplus Q'$ . Since  $X \oplus P \oplus P' \leq_{\deg} X \oplus Q \oplus Q'$ , we obtain  $\dim_k(P \oplus P') = \dim_k(Q \oplus Q')$ , which implies

$$P \oplus P' \cong Q \oplus Q'.$$

Since  $X \oplus P \leq_{\deg} Y \oplus Q$  and  $Y \oplus P' \leq_{\deg} X \oplus Q'$ , we have

$$X \oplus P \oplus P' \leq_{\deg} Y \oplus Q \oplus P'$$

and

$$Y \oplus P' \oplus Q \leq_{\deg} X \oplus Q' \oplus Q \cong X \oplus P \oplus P'.$$

From the anti-symmetry of the degeneration order  $\leq_{\deg}$ , we know that

$$X \oplus P \oplus P' \cong Y \oplus Q \oplus P'.$$

Hence we have  $X \cong Y$  in  $A\text{-}\underline{\mathbf{mod}}$ . This shows that  $\leq_{\text{st}}$  is anti-symmetric on the set of isomorphism classes of  $A\text{-}\underline{\mathbf{mod}}$ . In order to prove transitivity of  $\leq_{\text{st}}$ , let

$$X \leq_{\text{st}} Y$$

and

$$Y \leq_{\text{st}} Z$$

in  $A\text{-}\underline{\mathbf{mod}}$ . Then there exist projective  $A$ -modules  $P, Q, R$  and  $S$  such that

$$X \oplus P \leq_{\deg} Y \oplus Q$$

and

$$Y \oplus R \leq_{\deg} Z \oplus S.$$

Hence we have

$$X \oplus P \oplus R \leq_{\deg} Y \oplus Q \oplus R$$

and

$$Y \oplus Q \oplus R \leq_{\deg} Z \oplus Q \oplus S.$$

From the transitivity of  $\leq_{\deg}$  it follows that

$$X \oplus P \oplus R \leq_{\deg} Z \oplus Q \oplus S.$$

Hence  $X \leq_{\text{st}} Z$  in  $A\text{-}\underline{\mathbf{mod}}$ . Therefore the transitivity of  $\leq_{\text{st}}$  holds. ■

Next we define the triangle relation  $\leq_{\Delta}$  for a (left) triangulated category. For the concept of left triangulated categories we refer to [BeMa] and [KeVo].

**Definition 2.6** ([Yosh2], [JSZ2]). Let  $\mathcal{C}$  be a (left) triangulated category and  $X, Y \in \mathcal{C}$ , then we say that  $X \leq_{\Delta} Y$  if there exist  $Z \in \mathcal{C}$  and an exact triangle in  $\mathcal{C}$ ,

$$Z[-1] \rightarrow Y \rightarrow X \oplus Z \rightarrow Z.$$

**Remark 2.7.** As well-known,  $A\text{-mod}$  has a left triangulated structure with the syzygy functor  $\Omega_A$  as the translation functor (cf. [BeMa]). Hence, we can consider the triangle relation  $\leq_{\Delta}$  in  $A\text{-mod}$ . Next we will prove that  $\leq_{\Delta}$  coincides with  $\leq_{\text{st}}$  in  $A\text{-mod}$ .

**Lemma 2.8.** *Let  $A$  be a finite dimensional  $k$ -algebra and let  $X, Y \in A\text{-mod}$ . Then  $X \leq_{\text{st}} Y$  if and only if  $X \leq_{\Delta} Y$  in  $A\text{-mod}$ .*

*Proof.* Suppose that  $X \leq_{\text{st}} Y$  in  $A\text{-mod}$ . Then there exist projective  $A$ -modules  $P$  and  $Q$  such that  $X \oplus P \leq_{\text{deg}} Y \oplus Q$ . By Definition 2.1, there is an exact sequence in  $A\text{-mod}$ ,

$$0 \rightarrow Y \oplus Q \rightarrow X \oplus P \oplus Z \rightarrow Z \rightarrow 0.$$

This induces the following exact triangle in  $A\text{-mod}$ ,

$$\Omega_A(Z) \rightarrow Y \rightarrow X \oplus Z \rightarrow Z.$$

Hence  $X \leq_{\Delta} Y$ .

Conversely, assume that  $X \leq_{\Delta} Y$ . Then there exist an  $A$ -module  $Z$  and an exact triangle,

$$\Omega_A(Z) \rightarrow Y \rightarrow X \oplus Z \rightarrow Z.$$

By the construction of the left triangulated structure in  $A\text{-mod}$ , we know that there exist projective  $A$ -modules  $P, Q$  and  $R$  and an exact sequence in  $A\text{-mod}$ ,

$$0 \rightarrow Y \oplus P \rightarrow X \oplus Z \oplus Q \rightarrow Z \oplus R \rightarrow 0.$$

Consider the following commutative diagram,

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \uparrow & & \uparrow & \\ & & & Z & \xlongequal{\quad} & Z & \\ & & & \uparrow & & \uparrow & \\ & & & \alpha & & & \\ 0 & \longrightarrow & Y \oplus P & \longrightarrow & X \oplus Z \oplus Q & \longrightarrow & Z \oplus R \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow & \\ 0 & \longrightarrow & Y \oplus P & \longrightarrow & \ker(\alpha) & \longrightarrow & R \longrightarrow 0 \\ & & & & \uparrow & & \uparrow & \\ & & & & 0 & & 0 & \end{array}$$

where  $\alpha$  is the composition of  $X \oplus Z \oplus Q \rightarrow Z \oplus R \rightarrow Z$ . Since  $R$  is projective, the bottom row sequence splits, hence  $\ker \alpha \cong Y \oplus P \oplus R$  and we obtain the exact sequence,

$$0 \longrightarrow Y \oplus P \oplus R \longrightarrow X \oplus Q \oplus Z \xrightarrow{\alpha} Z \longrightarrow 0.$$

Therefore  $X \oplus Q \leq_{\text{deg}} Y \oplus P \oplus R$ , hence  $X \leq_{\text{st}} Y$  in  $A\text{-mod}$ . ■

**Lemma 2.9.** *Let  $A$  be a finite dimensional  $k$ -algebra. If  $X \leq_{\text{st}} Y$ , then  $\Omega_A(X) \leq_{\text{st}} \Omega_A(Y)$ .*

*Proof.* From Lemma 2.8 it follows that  $X \leq_{\text{st}} Y$  if and only if  $X \leq_{\Delta} Y$  in  $A\text{-}\underline{\text{mod}}$ . Hence we have the following exact triangle in  $A\text{-}\underline{\text{mod}}$ ,

$$\Omega_A(Z) \rightarrow Y \rightarrow X \oplus Z \rightarrow Z.$$

The Axioms of (left) triangulated categories imply that there is the following exact triangle,

$$\Omega_A^2(Z) \rightarrow \Omega_A(Y) \rightarrow \Omega_A(X) \oplus \Omega_A(Z) \rightarrow \Omega_A(Z).$$

Therefore  $\Omega_A(X) \leq_{\Delta} \Omega_A(Y)$ , hence  $\Omega_A(X) \leq_{\text{st}} \Omega_A(Y)$  using again Lemma 2.8. ■

### 3 Stable categories and stabilization

We recall some notions about the stabilization of stable categories. For details, we refer to [Bel].

**Definition 3.1.** Let  $(\mathcal{C}, \Omega, \Delta)$  be a left triangulated category. The stabilization of  $\mathcal{C}$  is a pair  $(\iota, \mathcal{S}(\mathcal{C}))$ , where  $\mathcal{S}(\mathcal{C})$  is a triangulated category and  $\iota : \mathcal{C} \rightarrow \mathcal{S}(\mathcal{C})$  is an exact functor, called the stabilization functor, such that for any exact functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  to a triangulated category  $\mathcal{D}$ , there exists a unique exact functor  $F^* : \mathcal{S}(\mathcal{C}) \rightarrow \mathcal{D}$  such that  $F^* \iota = F$ .

We recall the construction of  $\mathcal{S}(\mathcal{C})$  (cf. [Bel], [Hel], [KeVo]). An object of  $\mathcal{S}(\mathcal{C})$  is a pair  $(X, m)$  where  $X \in \mathcal{C}$  and  $m \in \mathbb{Z}$ .

$$\text{Hom}_{\mathcal{S}(\mathcal{C})}((X, m), (Y, n)) := \varinjlim_{k \geq m, n} \text{Hom}_{\mathcal{C}}(\Omega^{k-m} X, \Omega^{k-n} Y).$$

Then  $\mathcal{S}(\mathcal{C})$  is a triangulated category with the translation functor:  $\tilde{\Omega} : \mathcal{S}(\mathcal{C}) \rightarrow \mathcal{S}(\mathcal{C})$  defined as follows:  $\tilde{\Omega}(X, m) = (X, m - 1)$ .

$$\tilde{\Omega}(Z, l) \rightarrow (X, m) \rightarrow (Y, n) \rightarrow (Z, l)$$

is an exact triangle in  $\mathcal{S}(\mathcal{C})$  if and only if there exist  $k \in 2\mathbb{Z}$  and a triangle, which represents the above triangle,

$$\Omega(\Omega^{k-l}(Z)) \rightarrow \Omega^{k-m}(X) \rightarrow \Omega^{k-n}(Y) \rightarrow \Omega^{k-l}(Z)$$

in  $\mathcal{C}$ .

**Theorem 3.2** (Corollary 3.9. [Bel]). *Let  $A$  be a finite-dimensional  $k$ -algebra, then there exists a triangle equivalence*

$$\mathcal{S}(A\text{-}\underline{\text{mod}}) \cong \mathcal{D}_{\text{sg}}(A).$$

**Definition 3.3.** Let  $A$  be a finite dimensional  $k$ -algebra. Let  $(X, m), (Y, n) \in \mathcal{S}(A\text{-}\underline{\text{mod}})$ . We say that  $(Y, n)$  is a quasi-stable degeneration of  $(X, m)$  (denote by  $(X, m) \leq_{\text{qst}} (Y, n)$ ) if and only if there exists  $k \in \mathbb{N}$  such that  $\Omega^{k-m}(X) \leq_{\text{st}} \Omega^{k-n}(Y)$  in  $A\text{-}\underline{\text{mod}}$ .

**Remark 3.4.** Since  $\mathcal{S}(A\text{-}\underline{\text{mod}})$  is a triangulated category, there is the triangle relation  $\leq_{\Delta}$  (cf. Definition 2.6) in  $\mathcal{S}(A\text{-}\underline{\text{mod}})$ . Next we will show that these two relations  $\leq_{\text{qst}}$  and  $\leq_{\Delta}$  in  $\mathcal{S}(A\text{-}\underline{\text{mod}})$  coincide.

**Proposition 3.5.** *Let  $A$  be a finite dimensional  $k$ -algebra, Then in  $\mathcal{S}(A\text{-}\underline{\text{mod}})$  we have that  $(X, m) \leq_{\text{qst}} (Y, n)$  if and only if  $(X, m) \leq_{\Delta} (Y, n)$ .*

*Proof.* If  $(X, m) \leq_{\text{qst}} (Y, n)$ , then by Definition 3.3, there exists  $k \in \mathbb{N}$  such that  $\Omega^{k-m}(X) \leq_{\text{st}} \Omega^{k-n}(Y)$  in  $A\text{-}\underline{\text{mod}}$ , which means that there exist an  $A$ -module  $Z$  and an exact triangle in  $A\text{-}\underline{\text{mod}}$

$$\Omega(Z) \rightarrow \Omega^{k-n}(Y) \rightarrow \Omega^{k-m}(X) \oplus Z \rightarrow Z.$$

So we have a triangle in  $\mathcal{S}(A\text{-}\underline{\text{mod}})$

$$(Z, -1) \rightarrow (\Omega^{k-n}(Y), 0) \rightarrow (\Omega^{k-m}(X), 0) \oplus (Z, 0) \rightarrow (Z, 0),$$

which is isomorphic to

$$(Z, -1) \rightarrow (Y, n - k) \rightarrow (X, m - k) \oplus (Z, 0) \rightarrow (Z, 0).$$

So  $(X, m - k) \leq_{\Delta} (Y, n - k)$ , hence  $(X, m) \leq_{\Delta} (Y, n)$ .

Conversely, suppose that  $(X, m) \leq_{\Delta} (Y, n)$ . Then there exist  $(Z, r) \in \mathcal{S}(A\text{-}\underline{\text{mod}})$  and an exact triangle,

$$(Z, r - 1) \rightarrow (Y, n) \rightarrow (X, m) \oplus (Z, r) \rightarrow (Z, r),$$

that is, there exist  $k \in 2\mathbb{N}$  and an exact triangle in  $A\text{-}\underline{\text{mod}}$ ,

$$\Omega^{k-r+1}(Z) \rightarrow \Omega^{k-n}(Y) \rightarrow \Omega^{k-m}(X) \oplus \Omega^{k-r}(Z) \rightarrow \Omega^{k-r}(Z).$$

Hence  $\Omega^{k-m}(X) \leq_{\text{st}} \Omega^{k-n}(Y)$ , so  $(X, m) \leq_{\text{qst}} (Y, n)$ . ■

Now let us prove the main theorem (cf. Theorem 3.7). Before we come to the proof of Theorem 3.7, we need the following lemma.

**Lemma 3.6.** *Let  $A$  be a finite dimensional  $k$ -algebra. Then the quasi-stable degeneration relation  $\leq_{\text{qst}}$  in  $\mathcal{S}(A\text{-}\underline{\text{mod}})$  is a partial order on the set of isomorphism classes of objects in  $\mathcal{S}(A\text{-}\underline{\text{mod}})$ .*

*Proof.* The reflexivity is inherited from the stable degeneration in  $A\text{-}\underline{\text{mod}}$ . For transitivity, let  $(X, m) \leq_{\text{qst}} (Y, n)$  and  $(Y, n) \leq_{\text{qst}} (Z, l)$ , then there exist  $k_1, k_2 \in \mathbb{N}$  such that

$$\Omega^{k_1-m}(X) \leq_{\text{st}} \Omega^{k_1-n}(Y)$$

and

$$\Omega^{k_2-n}(Y) \leq_{\text{st}} \Omega^{k_2-l}(Z).$$

From Lemma 2.9, we have that

$$\Omega^{k_1+k_2-m}(X) \leq_{\text{st}} \Omega^{k_1+k_2-n}(Y)$$

and

$$\Omega^{k_1+k_2-n}(Y) \leq_{\text{st}} \Omega^{k_1+k_2-l}(Z).$$

By Lemma 2.5, we know that the stable degeneration  $\leq_{\text{st}}$  in  $A\text{-}\underline{\text{mod}}$  is a partial order, hence

$$\Omega^{k_1+k_2-m}(X) \leq_{\text{st}} \Omega^{k_1+k_2-l}(Z).$$

So  $(X, m) \leq_{\text{qst}} (Z, l)$ , the transitivity holds in  $\mathcal{S}(A\text{-}\underline{\text{mod}})$ . Similarly, we can show that the anti-symmetric. ■

**Theorem 3.7.** *Let  $A$  be a finite dimensional  $k$ -algebra, then the triangle order  $\leq_{\Delta}$  is a partial order on the set of isomorphism classes of  $\mathcal{D}_{\text{sg}}(A)$ .*

*Proof.* From Theorem 3.2, we know that  $\mathcal{D}_{\text{sg}}(A) \cong \mathcal{S}(A\text{-}\underline{\text{mod}})$  as triangulated categories. Hence it's sufficient to show that the triangle order  $\leq_{\Delta}$  in  $\mathcal{S}(A\text{-}\underline{\text{mod}})$  is a partial order. But this follows from Proposition 3.5 and Lemma 3.6. Therefore we have shown our result. ■

## References

- [Buch] Ragnar Olaf Buchweitz, *Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings*, manuscript Universität Hannover 1986.
- [Bel] Apostolos Beligiannis, *The homological theory of contravariantly finite subcategories: Auslander-Buchweitz contexts, Gorenstein categories and (co-) stabilization* Communications in Algebra, **28** (10), 4547-4596 (2000).
- [BeMa] Apostolos Beligiannis and Nikolaos Marmaridis, *Left triangulated categories arising from contravariantly finite subcategories*, Communications in Algebra **22** (1994), no. 12, 5021-5036.
- [Chen] Xiao-Wu Chen, *The singular category of an algebra with radical square zero*, Documenta Mathematica **16** (2011), no. 2-3, 299-212.
- [Hel] Alex Heller, *Stable homotopy categories*, Bulletin of the American Mathematical Society **74** (1968), 28-63.
- [JSZ1] Bernt Tore Jensen, Xiuping Su and Alexander Zimmermann, *Degenerations for derived categories*, Journal of Pure and Applied Algebra **198** (2005) 281-295.
- [JSZ2] Bernt Tore Jensen, Xiuping Su and Alexander Zimmermann, *Degeneration-like orders for triangulated categories*, Journal of Algebra and its Applications **4** (2005) 587-597.
- [KeVo] Bernhard Keller and Dieter Vossieck, *Sous les catégories dérivées*, Comptes Rendus de l'Académie des Sciences Paris, Série I Mathématique **305** (6) (1987) 225-228.
- [Lein] Tom Leinster, *The bijection between projective indecomposable and simple modules*, arXiv:1410.3671 [math.RA].



- [Orl] Dmitri Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Volume II, 503531, Progress in Mathematics, 270, Birkhuser Boston, Inc., Boston, MA, 2009.
- [Rie] Christine Riedtmann, *Degenerations for representations of quivers with relations*, Annales de l'École Normale Supérieure **19** (1986) 275-301.
- [SaZi] Manuel Saorin and Alexander Zimmermann, *An axiomatic approach for degenerations in triangulated categories*, Preprint (2014).
- [Yosh1] Yuji Yoshino, *On degeneration of Cohen-Macaulay modules*, Journal of Algebra **248** (2002) 271-290.
- [Yosh2] Yuji Yoshino, *On degeneration of modules*, Journal of Algebra **278** (2004) 217-226.
- [Yosh3] Yuji Yoshino, *Stable degeneration of Cohen-Macaulay modules*, Journal of Algebra **332** (2011) 500-521.
- [ZhZi] Guo-Dong Zhou and Alexander Zimmermann, *On Singular Equivalence of Morita Type*, Journal of Algebra **385** (2013) 64-79.
- [Zim] Alexander Zimmermann, *Representation Theory: A Homological Algebra Point of View*, Springer Verlag London, 2014.
- [Zwa1] Grzegorz Zwara, *A degeneration-like order for modules*, Archiv der Mathematik **71** (1998) 437-444.
- [Zwa2] Grzegorz Zwara, *Degenerations of finite dimension modules are given by extensions*, Compositio Mathematica **121** (2000) 205-218.